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A solution to the SU(n) external state labelling problem based upon a U($n-1, n-1$) group: II. Recursion relations for the SU(2) and SU(3) Wigner coefficients

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Abstract. The new solution to the SU(n) external state labelling problem, proposed in the first paper of this series, is analysed in detail in the SU(3) case in terms of the two complementary chains U(3) × U(3) ⊃ U(3) and U(2, 2) ⊃ U(2) × U(2). The classification of SU(3) coupled states is completed by the labels of an intermediate U(2) irreducible representation $h^s = [h_1^s h_2^s]$, directly related to King's branching rule for U(3) × U(3) ⊃ U(3). For pedagogical purposes, the SU(2) coupled state construction, based upon the complementary chains U(2) × U(2) ⊃ U(2) and U(1, 1) ⊃ U(1) × U(1), is considered first. For both SU(2) and SU(3), it is proved that, in addition to the standard recursion relations, the corresponding Wigner coefficients satisfy recursion relations of a different type, arising from the action of the U(1, 1) or U(2, 2) complementary group generators. A detailed example shows that such complementary recursion relations are quite useful for numerical purposes.

1. Introduction

The purpose of this series of papers is to present a new solution to the SU(n) external state labelling problem or, equivalently, to the internal state labelling problem for the chain U(n) × U(n) ⊃ U(n) when ($n-1$)-row U(n) irreducible representations (irreps) are considered. Such a solution is based upon the complementarity relationship (Moshinsky and Quesne 1970, Howe 1979) between the chains U(n) × U(n) ⊃ U(n) and U($n-1, n-1$) ⊃ U($n-1$) × U($n-1$) (Kashiwara and Vergne 1978, King and Wybourne 1985, Quesne 1986b).

The first paper in this series (hereafter referred to as I and whose equations will be subsequently quoted by their number preceded by I) was devoted to a presentation of the general solution and to the demonstration of its interesting group theoretical properties (Quesne 1987a). It indeed directly reflects the operation of King's U(n) × U(n) ⊃ U(n) branching rule (1970, 1971), which is just a reformulation of the well known Littlewood–Richardson rule (1934), valid for mixed U(n) irreps (Flores 1967, Flores and Moshinsky 1967, King 1970).

Since the proposed solution uses SU($n-1$) Wigner coefficients, it is based upon a recursive procedure. Hence, the first step in an explicit state construction is a detailed study of the SU(3) case, for which the complementary chains are U(3) × U(3) ⊃ U(3)

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and $U(2,2) \supset U(2) \times U(2)$. Such is the purpose of the present paper. Since much insight can be gained by first considering the complementary chains $U(2) \times U(2) \supset U(2)$ and $U(1,1) \supset U(1) \times U(1)$, for which there is no missing label, we shall actually begin with a review of the SU(2) state construction. For both SU(2) and SU(3), we shall prove that in addition to the standard recursion relations, the corresponding Wigner coefficients (WC) satisfy recursion relations of a different type. By working out a detailed example, we shall also show that the present solution to the state labelling problem is not only interesting from a group theoretical viewpoint, but also quite useful for practical purposes.

In § 2, we review the construction of SU(2) WC based upon U(1, 1), and in § 3, we apply it to derive recursion relations. In §§ 4 and 5, the corresponding problems are solved for SU(3). In § 6, we describe various procedures for orthonormalising the SU(3) WC. In § 7, we work out a detailed example. Finally, § 8 contains the conclusion. Except when stated otherwise, the notations are the same as in I.

2. SU(2) Wigner coefficients in terms of U(1, 1)

The purpose of the present section is to apply the results of I to construct SU(2) coupled states, and thence SU(2) WC.

SU(2) coupled states are states classified according to the group chain

$$\begin{array}{cccc} U(2) \times & U(2) \supset & U(2) \supset & U(1) \\ h & h'^* & \kappa & \mu \end{array} \tag{2.1}$$

where below each group we have indicated the labels characterising its irreps. Here h, h' and κ are shorthand for $[h0]$, $[h'0]$ and $[k, -k']$, respectively, where h, h', k and k' are non-negative integers, and μ is an integer such that $-k' \leq \mu \leq k$. Throughout this paper, an asterisk on an irrep label is used to denote the contragredient irrep, i.e., if $f = [f_1 f_2 \ldots f_n]$, then $f^* = [-f_n, \ldots, -f_2, -f_1]$, hence $h'^* = [0, -h']$. According to King's branching rule (1970), given in equation (I2.12), an irrep κ is contained in the product $h \times h'^*$ provided that

$$h - k = h' - k' = h^{\backprime} \geq 0. \tag{2.2}$$

As is well known, there is no missing label ω in the chain (2.1).

Following I, we now consider the complementary chain

$$\begin{array}{ccc} U(1,1) \supset & U(1) \times & U(1) \\ [k; k'] & h & h' \end{array} \tag{2.3}$$

where below each group we have again indicated the labels characterising its irreps. All the SU(2) coupled states corresponding to given irreps κ and μ of U(2) and U(1) and to the various irreps $h \times h'^*$ of $U(2) \times U(2)$ containing κ, i.e. satisfying condition (2.2), belong to a single infinite-dimensional irrep of U(1, 1), specified by $[k; k']$. Hence, they can be written as

$$\left| \begin{matrix} [k; k'] & & h & h'^* \\ h & h' & ; & \kappa \\ & & & \mu \end{matrix} \right\rangle = |(hh'^*)\kappa\mu\rangle \tag{2.4}$$

where on the left-hand side we use the notation of equation (I3.15), and on the right-hand side the simplified notation of the present paper. Their explicit form is given by

$$|(hh'^*)\kappa\mu\rangle = AP^{h^s \times h^s}(D^\dagger_{12})|(kk'^*)\kappa\mu\rangle \tag{2.5}$$

where

$$P^{h^s \times h^s}(D^\dagger_{12}) = (D^\dagger_{12})^{h^s} \tag{2.6}$$

and

$$A = [(k+k'+1)!]^{1/2}[h^s!(h^s+k+k'+1)!]^{-1/2}. \tag{2.7}$$

The latter can be obtained either by direct calculation or by using a coherent state technique (Quesne 1987b). In the following, we shall not need the explicit form of the states $|(kk'^*)\kappa\mu\rangle$, which could be deduced from equations (I5.1)-(I5.3). Such states will be referred to as stretched ones, since they correspond to the stretched product of the irreps k and k'^*.

The SU(2) WC are the transformation coefficients between the states classified according to the chains (2.1) and

$$\begin{array}{cccc} \mathrm{U}(2) \times & \mathrm{U}(2) \supset & \mathrm{U}(1) \times & \mathrm{U}(1) \\ h & h'^* & q & q'^* \end{array} \tag{2.8}$$

where $\mathrm{U}(1)\times\mathrm{U}(1)$ is generated by the operators $\mathscr{P}'_{11}$ and $\mathscr{P}''_{11}$ (cf equations (I2.3) and (I2.7)). The states transforming under (2.8) can be written as

$$|hq, h'^*q'^*\rangle = [q!(h-q)!q'!(h'-q')!]^{-1/2}(\eta_{11})^q(\eta_{12})^{h-q}(\eta_{21})^{q'}(-\eta_{22})^{h'-q'}|0\rangle \tag{2.9}$$

where we have taken into account that η_{11}, η_{12} form a contravariant vector, whereas η_{21}, η_{22} form a covariant one.

To compare the present construction of SU(2) WC with standard ones, let us rewrite the chains (2.1), (2.3) and (2.8) in terms of unimodular groups as follows:

$$\begin{array}{cccc} \mathrm{SU}(2) \times & \mathrm{SU}(2) \supset & \mathrm{SU}(2) \supset & \mathrm{U}(1) \\ j & j' & J & M \end{array} \tag{2.10}$$

$$\begin{array}{cc} \mathrm{SU}(1,1) \supset & \mathrm{U}(1) \\ K & Q \end{array} \tag{2.11}$$

$$\begin{array}{cccc} \mathrm{SU}(2) \times & \mathrm{SU}(2) \supset & \mathrm{U}(1) \times & \mathrm{U}(1). \\ j & j' & m & m' \end{array} \tag{2.12}$$

The generators of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ group are defined in terms of $\mathscr{P}'_{st}$ and $\mathscr{P}''_{st}$, $s, t = 1, 2$, by the relations

$$\begin{aligned} &j_+ = \mathscr{P}'_{12} = \eta_{11}\xi_{12} \qquad j_- = \mathscr{P}'_{21} = \eta_{12}\xi_{11} \\ &j_0 = \tfrac{1}{2}(\mathscr{P}'_{11} - \mathscr{P}'_{22}) = \tfrac{1}{2}(\eta_{11}\xi_{11} - \eta_{12}\xi_{12}) \qquad j'_+ = \mathscr{P}''_{12} = -\eta_{22}\xi_{21} \\ &j'_- = \mathscr{P}''_{21} = -\eta_{21}\xi_{22} \qquad j'_0 = \tfrac{1}{2}(\mathscr{P}''_{11} - \mathscr{P}''_{22}) = \tfrac{1}{2}(\eta_{22}\xi_{22} - \eta_{21}\xi_{21}) \end{aligned} \tag{2.13}$$

while those of its $\mathrm{U}(1)\times\mathrm{U}(1)$, SU(2) and U(1) subgroups are j_0, j'_0, $\boldsymbol{J} = \boldsymbol{j} + \boldsymbol{j}'$ and J_0, respectively. On the other hand, the SU(1, 1) generators are

$$\begin{aligned} &K_+ = D^\dagger_{12} = \sum_{s=1}^{2} \eta_{1s}\eta_{2s} \qquad K_- = D_{12} = \sum_{s=1}^{2} \xi_{1s}\xi_{2s} \\ &K_0 = \tfrac{1}{2}(E_{11} + E_{22}) = \tfrac{1}{2}\sum_{i,s=1}^{2}(\eta_{is}\xi_{is} + \tfrac{1}{2}) \end{aligned} \tag{2.14}$$

where K_0 generates the U(1) subgroup. Note that the operator

$$\mathbb{G}_1 = \mathcal{P}_{11} + \mathcal{P}_{22} = E_{11} - E_{22} = \sum_{s=1}^{2} (\eta_{1s}\xi_{1s} - \eta_{2s}\xi_{2s}) \tag{2.15}$$

which is the common first-order Casimir operator of U(2) and U(1, 1), commutes with both sets of operators J_+, J_-, J_0, and K_+, K_-, K_0. Its eigenvalue is $\rho = 2(j-j')$, while those of $\boldsymbol{K}^2 = -K_+K_- + K_0(K_0-1)$ and K_0 are $K(K-1)$, where $K = J+1$ and $Q = j+j'+1$, respectively. The quantum numbers specifying the irreps in equations (2.10) and (2.12) are related to the labels in equations (2.1) and (2.8) as follows:

$$\begin{aligned} &h = 2j \qquad h' = 2j' \qquad k = j-j'+J \qquad k' = j'-j+J \\ &\mu = j-j'+M \qquad q = j+m \qquad q' = j'-m'. \end{aligned} \tag{2.16}$$

In the SU(2) notation, the states (2.5) and (2.9) can be rewritten as

$$|jj'JM\rangle = [(2J+1)!]^{1/2}[(j+j'-J)!(j+j'+J+1)!]^{-1/2}(K_+)^{j+j'-J}|\bar{j}\bar{j}'JM\rangle \tag{2.17}$$

and

$$\begin{aligned} |jm, j'm'\rangle = {} & [(j+m)!(j-m)!(j'+m')!(j'-m')!]^{-1/2} \\ & \times (\eta_{11})^{j+m}(\eta_{12})^{j-m}(\eta_{21})^{j'-m'}(-\eta_{22})^{j'+m'}|0\rangle \end{aligned} \tag{2.18}$$

where $\bar{j} = (j-j'+J)/2$ and $\bar{j}' = (j'-j+J)/2$. Hence all the coupled states with total angular momentum J and projection M can be obtained from the stretched state $|\bar{j}\bar{j}'JM\rangle$, for which $\bar{j}+\bar{j}' = J$, by successive applications of K_+.

Equation (2.18) coincides with equation (3.30) of Schwinger (1965), except for the fact that the realisation of K_+ (denoted $\mathcal{K}_+$ by Schwinger) is different. One can actually go from Schwinger's analysis to the present one by the replacement of $a_+^\dagger$, $a_-^\dagger$, $b_+^\dagger$, $b_-^\dagger$ by η_{11}, η_{12}, $-\eta_{22}$, η_{21}, and a similar substitution for the annihilation operators. Such a transformation, quite immaterial for SU(2), has however far-reaching consequences when going to higher SU(n) groups. The extension of the present operators K_+, K_-, K_0 is indeed straightforward, whereas Schwinger's ones $\mathcal{K}_+$, $\mathcal{K}_-$, $\mathcal{K}_0$ cannot be directly generalised. Together with three additional operators $\mathcal{J}_+$, $\mathcal{J}_-$, $\mathcal{J}_0$, also considered by Schwinger, the latter indeed form the generators of an SU(1, 1) × SU(2) group, locally isomorphic to SO*(4). Since SO*(4) is complementary with respect to USp(2) (Quesne 1985 and references quoted therein), Schwinger's treatment depends on the local isomorphism between USp(2) and SU(2), which cannot be extended to higher n values.

In the next section, we shall use the present construction of SU(2) WC to derive some recursion relations.

3. Recursion relations for the SU(2) Wigner coefficients

It is well known that by considering the matrix elements of the SU(2) generators $\mathcal{P}_{12} = J_+$ and $\mathcal{P}_{21} = J_-$ between a coupled state and an uncoupled one, one finds two recursion relations for the SU(2) WC (see e.g. Edmonds 1957):

$$\begin{aligned} & f_\pm(J, M)\langle jm, j'm'|JM\rangle \\ & \quad = f_\pm(j, m)\langle jm\pm 1, j'm'|JM\pm 1\rangle + f_\pm(j', m')\langle jm, j'm'\pm 1|JM\pm 1\rangle \end{aligned} \tag{3.1}$$

where

$$f_\pm(\alpha, \beta) = [(\alpha \mp \beta)(\alpha \pm \beta + 1)]^{1/2}. \tag{3.2}$$

When used in conjunction with the orthogonality properties of WC and the Condon-Shortley phase convention (1935), they lead to Racah's formula (1942) for the SU(2) WC.

The complementary nature of the SU(2) coupled states (2.17) leads to another set of recursion relations, which will be referred to as the complementary ones, as opposed to the former set, referred to as the standard ones. The complementary recursion relations are obtained by considering the matrix elements $\langle jm, j'm'|K_{\pm}|j\mp\frac{1}{2}j'\mp\frac{1}{2}JM\rangle$ of the SU(1, 1) generators K_+ and K_- between a coupled state and an uncoupled one. From equation (2.17) and the commutation relation

$$[K_-, K_+] = 2K_0 \tag{3.3}$$

we readily obtain

$$K_{\pm}|j\mp\tfrac{1}{2}j'\mp\tfrac{1}{2}JM\rangle = [(j+j'-J+\tfrac{1}{2}\mp\tfrac{1}{2})(j+j'+J+\tfrac{3}{2}\mp\tfrac{1}{2})]^{1/2}|jj'JM\rangle. \tag{3.4}$$

On the other hand, the action of $K_{\pm}$ on the bra $\langle jm, j'm'|$ can be easily determined from equation (2.14) and the known action of η_{is} and ξ_{is} on the states (2.18). The resulting relations are

$$\begin{aligned} F_{\pm}(j+j'-J, j+j'+J+1)\langle jm, j'm'|JM\rangle \\ = F_{\pm}(j+m, j'-m')\langle j\mp\tfrac{1}{2}\, m\mp\tfrac{1}{2}, j'\mp\tfrac{1}{2}\, m'\pm\tfrac{1}{2}|JM\rangle \\ - F_{\pm}(j-m, j'+m')\langle j\mp\tfrac{1}{2}\, m\pm\tfrac{1}{2}, j'\mp\tfrac{1}{2}\, m'\mp\tfrac{1}{2}|JM\rangle \end{aligned} \tag{3.5}$$

where

$$F_{\pm}(\alpha, \alpha') = [(\alpha+\tfrac{1}{2}\mp\tfrac{1}{2})(\alpha'+\tfrac{1}{2}\mp\tfrac{1}{2})]^{1/2}. \tag{3.6}$$

Contrary to the standard relations, where j, j' and J are kept fixed, the complementary ones are characterised by given values of $j-j'$, J and M. They were obtained by Bargmann (1962) by a generating function technique. However, the present derivation clearly exhibits their group theoretical foundation.

The complementary recursion relations can be iterated in the same way as the standard ones to give Racah's formula. By solving the first relation, corresponding to the upper signs in equation (3.5), we can indeed express $\langle jm, j'm'|JM\rangle$ as a linear combination of stretched WC $\langle \bar{j}\bar{m}, \bar{j}'\bar{m}'|JM\rangle$, for which $\bar{j} = (j-j'+J)/2$ and $\bar{j}' = (j'-j+J)/2$ add to J. From the second relation, corresponding to the lower signs in equation (3.5), the stretched WC can then be determined in terms of one of them, namely $\langle \bar{j}\, M-\bar{j}', \bar{j}'\,\bar{j}'|JM\rangle$ or $\langle \bar{j}\, -\bar{j}, \bar{j}'\, M+\bar{j}|JM\rangle$ according as $M \geq j'-j$ or $M < j'-j$. Finally, the latter is obtained from the orthogonality properties of WC and the assumption that stretched WC are positive. Since the latter is a consequence of the Condon-Shortley phase convention, the phase choices made for the coupled and uncoupled states in equations (2.17) and (2.18) are consistent with such a convention.

An interesting property of the complementary recursion relations is that they reveal the existence of those Regge symmetries (1958) which remain hidden when considering only the standard recursion relations. From the latter, only the twelve Regge symmetries resulting from the six permutations of the columns and the interchange of the first two rows of the Regge array

$$\begin{bmatrix} j+m & j'+m' & J-M \\ j-m & j'-m' & J+M \\ -j+j'+J & j-j'+J & j+j'-J \end{bmatrix} \tag{3.7}$$

are obvious. It is however clear that the WC $\langle jm, j'm'|JM\rangle$ and $\langle\frac{1}{2}(j+j'+M)\,\frac{1}{2}(j-j'+m-m'), \frac{1}{2}(j+j'-M)\,\frac{1}{2}(j-j'-m+m')|Jj-j'\rangle$ satisfy the same complementary recursion relations. Hence they can only differ by a coefficient dependent on $j-j'$, J and M (and independent of $j+j'$ and $m-m'$). From the WC orthogonality properties, it results that such a coefficient must reduce to a phase factor. For $j+j'=J$, both WC are stretched ones which are positive, so that the phase factor is actually equal to one. We conclude that

$$\langle jm, j'm'|JM\rangle = \langle\tfrac{1}{2}(j+j'+M)\,\tfrac{1}{2}(j-j'+m-m'), \tfrac{1}{2}(j+j'-M)\,\tfrac{1}{2}(j-j'-m+m')|Jj-j'\rangle \tag{3.8}$$

proving the symmetry of the WC under transposition of the Regge array (3.7). By combining the latter with the symmetries under permutations of the columns, we obtain the six symmetries under permutations of the rows, and thence the full 72-element symmetry group first encountered by Regge.

Before generalising the present analysis to SU(3) in §§ 4 and 5, it is worth noting that the second complementary recursion relation, corresponding to the lower signs in equation (3.5), has not been used in its full generality, but only to calculate stretched WC. For such a purpose, only the action of K_- on stretched states is needed, namely

$$K_-|\bar{j}\bar{j}'JM\rangle = 0 \qquad \text{whenever } \bar{j}+\bar{j}'=J. \tag{3.9}$$

This relation directly results from the coupled state definition (2.17) and does not make use of equation (3.3).

4. SU(3) Wigner coefficients in terms of U(2, 2)

SU(3) coupled states are states classified according to either the group chain

$$\begin{matrix} \mathrm{U}(3) & \times\,\mathrm{U}(3) \supset & \mathrm{U}(3) \supset & \mathrm{U}(2) \supset & \mathrm{U}(1) \\ h & h'^* & \kappa & \lambda=\lambda'^* & \mu \end{matrix} \tag{4.1}$$

or

$$\begin{matrix} \mathrm{U}(2,2) \supset & \mathrm{U}(2)\times & \mathrm{U}(2) \supset & \mathrm{U}(1)\times & \mathrm{U}(1). \\ [k;k'] & h & h' & f & f' \end{matrix} \tag{4.2}$$

Here $h=[h_1h_20]$ and $h'=[h_1'h_2'0]$ for U(3), $h=[h_1h_2]$ and $h'=[h_1'h_2']$ for U(2), $\kappa=[k_1, k_2-k_2', -k_1']$ and $[k;k']=[k_1k_2;k_1'k_2']$, where $h_1\geqslant h_2\geqslant 0$, $h_1'\geqslant h_2'\geqslant 0$, $k_1\geqslant k_2\geqslant 0$, $k_1'\geqslant k_2'\geqslant 0$, and either $k_2=0$ or $k_2'=0$. In the latter case, for the U(2) subgroup irreps we use the notation $\lambda=[\lambda_1\lambda_2]$ with $\lambda_1\geqslant\lambda_2$ and $\lambda_1\geqslant 0$, while in the former we employ $\lambda'^*=[-\lambda_2', -\lambda_1']$ with $\lambda_1'\geqslant\lambda_2'$ and $\lambda_1'\geqslant 0$. In the relations to be derived below , the λ notation is used throughout, thus implicitly assuming that $k_2'=0$. Whenever $k_2=0$, we only have to replace λ_1 and λ_2 by $-\lambda_2'$ and $-\lambda_1'$ respectively.

According to King's branching rule (1970), the multiplicity of κ in $h\times h'^*$ is equal to the number of U(2) irreps $h^s=[h_1^sh_2^s]$ such that h and h' are respectively contained in the products $k\times h^s$ and $k'\times h^s$, where $k=[k_1k_2]$ and $k'=[k_1'k_2']$. From I, it follows that the classification of states according to either chain (4.1) or (4.2) can be completed by $\omega=h^s$, where, since $h_1^s+h_2^s=h_1+h_2-k_1-k_2=h_1'+h_2'-k_1'-k_2'$, there is only one independent label, as it should be. By using notations similar to those of equation (2.4), such states can be written as

$$\left|\begin{matrix}[k;k'] \\ h^{s}hh' \\ ff'\end{matrix};\begin{matrix}hh'^{*} \\ h^{s}\kappa \\ \lambda \\ \mu\end{matrix}\right\rangle = |ff';(hh'^{*})h^{s}\kappa\lambda\mu\rangle$$

$$= A_{h^{s}}[P^{h^{s}\times h^{s}}(D^{\dagger}_{ij})\times|(kk'^{*})\kappa\lambda\mu\rangle]^{h\times h'}_{f\times f'} \tag{4.3}$$

where $A_{h^{s}}$ is a normalisation constant,

$$D^{\dagger}_{ij} = \sum_{s=1}^{3} \eta_{is}\eta_{js} \qquad i=1,2, j=3,4 \tag{4.4}$$

and the square bracket notation means that the U(2) × U(2) irreps $k\times k'$ and $h^{s}\times h^{s}$ are coupled to $h\times h'$.

Since in any case the states (4.3) are not orthogonal with respect to h^{s}, it is more convenient to suppress the normalisation constant in their definition and to work with the non-normalised states

$$|ff';(hh'^{*})h^{s}\kappa\lambda\mu\rangle\rangle = [P^{h^{s}\times h^{s}}(D^{\dagger}_{ij})\times|(kk'^{*})\kappa\lambda\mu\rangle]^{h\times h'}_{f\times f'} \tag{4.5}$$

whose detailed expression reads

$$\begin{aligned}|ff';(hh'^{*})h^{s}\kappa\lambda\mu\rangle\rangle = \sum_{ll'mm'} &\langle\tfrac{1}{2}(k_1-k_2)\ l-\tfrac{1}{2}(k_1+k_2),\tfrac{1}{2}(h^{s}_1-h^{s}_2)\\ &\times m-\tfrac{1}{2}(h^{s}_1+h^{s}_2)|\tfrac{1}{2}(h_1-h_2)\ f-\tfrac{1}{2}(h_1+h_2)\rangle\langle\tfrac{1}{2}(k'_1-k'_2)\ l'-\tfrac{1}{2}(k'_1+k'_2),\\ &\tfrac{1}{2}(h^{s}_1-h^{s}_2)\ m'-\tfrac{1}{2}(h^{s}_1+h^{s}_2)|\tfrac{1}{2}(h'_1-h'_2)\ f'-\tfrac{1}{2}(h'_1+h'_2)\rangle\\ &\times P^{h^{s}\times h^{s}}_{m\times m'}(D^{\dagger}_{ij})|ll';(kk'^{*})\kappa\lambda\mu\rangle\end{aligned} \tag{4.6}$$

where ⟨ , | ⟩ denotes an SU(2) WC. By using U(2) lowering operators (Nagel and Moshinsky 1965), the polynomials $P^{h^{s}\times h^{s}}_{m\times m'}(D^{\dagger}_{ij})$, transforming under the irreps $h^{s}\times h^{s}$ and $m\times m'$ of U(2) × U(2) and U(1) × U(1) respectively, are readily obtained from equation (I4.4) in the form

$$\begin{aligned}P^{h^{s}\times h^{s}}_{m\times m'}(D^{\dagger}_{ij}) = {}&[(h^{s}_1-m)!(m-h^{s}_2)!(h^{s}_1-m')!(m'-h^{s}_2)!]^{1/2}\\ &\times(D^{\dagger}_{12,34})^{h^{s}_2}\sum_{\nu}[(\nu-h^{s}_2)!(m-\nu)!(m'-\nu)!(h^{s}_1-m-m'+\nu)!]^{-1}\\ &\times(D^{\dagger}_{13})^{\nu-h^{s}_2}(D^{\dagger}_{14})^{m-\nu}(D^{\dagger}_{23})^{m'-\nu}(D^{\dagger}_{24})^{h^{s}_1-m-m'+\nu}\end{aligned} \tag{4.7}$$

where

$$D^{\dagger}_{12,34} = D^{\dagger}_{13}D^{\dagger}_{24} - D^{\dagger}_{14}D^{\dagger}_{23}. \tag{4.8}$$

The explicit expression of the stretched stated $|ll';(kk'^{*})\kappa\lambda\mu\rangle$ will not be needed in the following and will therefore not be given here. Those states (4.5) whose weight is highest in U(2) × U(2) are simply denoted by

$$|(hh'^{*})h^{s}\kappa\lambda\mu\rangle\rangle \equiv |h_1h'_1;(hh'^{*})h^{s}\kappa\lambda\mu\rangle\rangle. \tag{4.9}$$

SU(3) uncoupled states are states classified according to the group chain

$$\begin{matrix}\mathrm{U}(3)\times\mathrm{U}(3) \supset \mathrm{U}(2)\times\mathrm{U}(2) \supset \mathrm{U}(1)\times\mathrm{U}(1)\\ h\quad h'^{*}\qquad q\quad q'^{*}\qquad r\quad r'^{*}\end{matrix} \tag{4.10}$$

where $q=[q_1q_2]$, $q'=[q'_1q'_2]$, $q_1\geqslant q_2\geqslant 0$, $q'_1\geqslant q'_2\geqslant 0$, $r\geqslant 0$ and $r'\geqslant 0$. In analogy with equation (4.3), they can be written in either notation

$$\left|\begin{matrix} - & & h\,h'^* \\ h & h'; & q\,q'^* \\ f & f' & r\,r'^* \end{matrix}\right\rangle = |ff';\, hqr,\, h'^*q'^*r'^*\rangle. \tag{4.11}$$

Whenever their weight is highest in the $U(2)\times U(2)$ subgroup of $U(2,2)$ and in the first $U(3)$ group, and lowest in the second one, such states become

$$\begin{aligned} |h(\max),\, h'^*(\min)\rangle &\equiv |h_1h'_1;\, h(\max),\, h'^*(\min)\rangle \\ &= [(h_1-h_2+1)(h'_1-h'_2+1)]^{1/2}[(h_1+1)!\,h_2!\,(h'_1+1)!\,h'_2!]^{-1/2} \\ &\quad\times(\eta_{11})^{h_1-h_2}(\eta_{12,12})^{h_2}(\eta_{31})^{h'_1-h'_2}(-\eta_{34,12})^{h'_2}|0\rangle \end{aligned} \tag{4.12}$$

where

$$\eta_{ij,12}=\eta_{i1}\eta_{j2}-\eta_{i2}\eta_{j1}. \tag{4.13}$$

The remaining uncoupled states can be obtained from equation (4.12) by standard lowering and raising techniques (Nagel and Moshinsky 1965, Quesne 1986a). Their detailed form will however not be needed in the derivation of the recursion relations.

The SU(3) WC are the scalar products of the states (4.6) with the states (4.11)

$$\left\langle\begin{matrix} h & & h'^* \\ q & , & q'^* \\ r & & r'^* \end{matrix}\,\right|\left.\begin{matrix} \kappa & & \\ \lambda & ; & h^s \\ \mu & & \end{matrix}\right\rangle = \langle hqr,\, h'^*q'^*r'^*|(hh'^*)h^s\kappa\lambda\mu\rangle\!\rangle \tag{4.14}$$

and are independent of f and f', which are therefore assumed maximal. As usual, they factorise into an SU(2) WC and an SU(3) isoscalar factor or reduced Wigner coefficient (RWC) as follows:

$$\begin{aligned} &\left\langle\begin{matrix} h & & h'^* \\ q & , & q'^* \\ r & & r'^* \end{matrix}\,\right|\left.\begin{matrix} \kappa & & \\ \lambda & ; & h^s \\ \mu & & \end{matrix}\right\rangle \\ &\quad= \langle\tfrac{1}{2}(q_1-q_2)\ r-\tfrac{1}{2}(q_1+q_2),\tfrac{1}{2}(q'_1-q'_2)\ -r'+\tfrac{1}{2}(q'_1+q'_2)\,|\,\tfrac{1}{2}(\lambda_1-\lambda_2)\ \mu-\tfrac{1}{2}(\lambda_1+\lambda_2)\rangle \\ &\quad\quad\times\left\langle\begin{matrix} h & & h'^* \\ q & , & q'^* \end{matrix}\,\right|\left.\begin{matrix} \kappa & & \\ \lambda & ; & h^s \end{matrix}\right\rangle. \end{aligned} \tag{4.15}$$

In the next section, we shall derive the recursion relations satisfied by the SU(3) RWC

$$\left\langle\begin{matrix} h & & h'^* \\ q & , & q'^* \end{matrix}\,\right|\left.\begin{matrix} \kappa & & \\ \lambda & ; & h^s \end{matrix}\right\rangle = \langle hq,\, h'^*q'^*,\, \lambda\mu|(hh'^*)h^s\kappa\lambda\mu\rangle\!\rangle \tag{4.16}$$

which are the overlaps of the SU(3) coupled states (4.9) with the SU(2) coupled ones

$$\begin{aligned} |hq,\, h'^*q'^*,\, \lambda\mu\rangle = \sum_{rr'} &\langle\tfrac{1}{2}(q_1-q_2)\ r-\tfrac{1}{2}(q_1+q_2),\tfrac{1}{2}(q'_1-q'_2) \\ &-r'+\tfrac{1}{2}(q'_1+q'_2)|\tfrac{1}{2}(\lambda_1-\lambda_2)\ \mu-\tfrac{1}{2}(\lambda_1+\lambda_2)\rangle|hqr,\, h'^*q'^*r'^*\rangle \end{aligned} \tag{4.17}$$

and are independent of μ.

5. Recursion relations for the SU(3) reduced Wigner coefficients

By considering the matrix elements of $\mathscr{P}_{13}$, $\mathscr{P}_{23}$, $\mathscr{P}_{31}$ and $\mathscr{P}_{32}$ between an SU(3) coupled state and an SU(2) coupled one, one obtains the four standard recursion relations for the SU(3) RWC (see e.g. Hecht 1965, Draayer and Akiyama 1973)†,

$$f(\kappa, \lambda, \bar{\lambda})\left\langle \begin{matrix} h \\ q \end{matrix}, \begin{matrix} h'^* \\ q'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \lambda \end{matrix}; h^s \right\rangle$$
$$= \sum_{\bar{q}} U(q', \lambda, \bar{q}, \sigma; q, \bar{\lambda}) f(h, q, \bar{q}) \left\langle \begin{matrix} h \\ \bar{q} \end{matrix}, \begin{matrix} h'^* \\ q'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \bar{\lambda} \end{matrix}; h^s \right\rangle$$
$$+ \varepsilon \sum_{\bar{q}'} U(q, \lambda^*, \bar{q}', \sigma^*; q', \bar{\lambda}^*) f(h', q', \bar{q}') \left\langle \begin{matrix} h \\ q \end{matrix}, \begin{matrix} h'^* \\ \bar{q}'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \bar{\lambda} \end{matrix}; h^s \right\rangle \tag{5.1}$$

corresponding to $\sigma = [10]$, $\bar{\lambda} = [\lambda_1 + 1, \lambda_2]$, $[\lambda_1, \lambda_2 + 1]$, and $\sigma = [0, -1]$, $\bar{\lambda} = [\lambda_1 - 1, \lambda_2]$, $[\lambda_1, \lambda_2 - 1]$, respectively. In equation (5.1), the summations run over $\bar{q} = [q_1 \pm 1, q_2]$, $[q_1, q_2 \pm 1]$, and $\bar{q}' = [q_1' \mp 1, q_2']$, $[q_1', q_2' \mp 1]$, where the upper signs correspond to $\sigma = [10]$ and the lower ones to $\sigma = [0, -1]$; the U coefficients are U(2) recoupling coefficients, equivalent to SU(2) ones (Edmonds 1957) according to the relation

$$U(\alpha, \beta, \gamma, \delta; \varepsilon, \zeta)$$
$$= U(\tfrac{1}{2}(\alpha_1 - \alpha_2), \tfrac{1}{2}(\beta_1 - \beta_2), \tfrac{1}{2}(\gamma_1 - \gamma_2), \tfrac{1}{2}(\delta_1 - \delta_2); \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2), \tfrac{1}{2}(\zeta_1 - \zeta_2)) \tag{5.2}$$

the f coefficients are defined by

$$f(\alpha, \beta, \bar{\beta}) = (\beta_1 - \beta_2 + 1)^{-1/2} \begin{cases} g_1(\alpha, \beta) & \text{if } \bar{\beta} = [\beta_1 + 1, \beta_2] \\ g_2(\alpha, \beta) & \text{if } \bar{\beta} = [\beta_1, \beta_2 + 1] \\ g_1(\alpha, \bar{\beta}) & \text{if } \bar{\beta} = [\beta_1 - 1, \beta_2] \\ -g_2(\alpha, \bar{\beta}) & \text{if } \bar{\beta} = [\beta_1, \beta_2 - 1] \end{cases} \tag{5.3}$$

where

$$g_1(\alpha, \beta) = [(\alpha_1 - \beta_1)(\beta_1 - \alpha_2 + 1)(\beta_1 - \alpha_3 + 2)]^{1/2} \tag{5.4}$$

and

$$g_2(\alpha, \beta) = [(\alpha_1 - \beta_2 + 1)(\alpha_2 - \beta_2)(\beta_2 - \alpha_3 + 1)]^{1/2}. \tag{5.5}$$

Finally, ε is a phase factor given by

$$\varepsilon = (-1)^{(\lambda_1 - \lambda_2 - \bar{\lambda}_1 + \bar{\lambda}_2 - 1)/2}. \tag{5.6}$$

Note that not only h, h'^* and κ, but also the additional label h^s, are kept fixed. The latter, characterising the transformation properties of the U(3) scalar polynomials $P^{h^s \times h^{s'}}_{m \times m'}(D^\dagger_{ij})$ under U(2) × U(2), cannot be changed by the SU(3) generators used to derive the recursion relations.

The complementary recursion relations for the SU(3) RWC are obtained by considering the matrix elements of $D^\dagger_{ij}$ and D_{ij}, $i = 1, 2$, $j = 3, 4$, between an SU(3) coupled state and an SU(2) coupled one. Only those of $D^\dagger_{ij}$ assume a simple form and will be considered here in full generality. Following the discussion at the end of § 3, those of D_{ij} are only needed in the stretched state case, to which we shall therefore restrict ourselves.

† For the states classified according to U(3) ⊃ U(2) ⊃ U(1), we use the phase convention of Baird and Biedenharn (1963) and Nagel and Moshinsky (1965), which differs from that of Hecht (1965).

Let us begin with the calculation of the $D^\dagger_{ij}$ matrix elements. Our starting point is the set of two relations

$$P^{h^s\times h^s}_{m\times m'}(D^\dagger_{ij})=[P^{[10]\times[10]}(D^\dagger_{ij})\times P^{\bar h^s\times \bar h^s}(D^\dagger_{ij})]^{h^s\times h^s}_{m\times m'} \tag{5.7}$$

where $\bar h^s=[\bar h^s_1\bar h^s_2]$ is either $[h^s_1-1,h^s_2]$ or $[h^s_1,h^s_2-1]$. Here the U(2)×U(2) irreps $\bar h^s\times\bar h^s$ and $[10]\times[10]$ are coupled to $h^s\times h^s$, and the components $P^{[10]\times[10]}_{\mu\times\mu'}$, $\mu,\mu'=1$, 0, coincide with the $D^\dagger_{ij}$ operators, as follows from equation (4.7). The proof of equation (5.7) is by direct substitution of equation (4.7) for $m=m'=h^s_1$.

By introducing equation (5.7) into the SU(3) coupled state definition (4.6) and by changing the coupling order, we obtain the set of two relations

$$|(hh'^*)h^s\kappa\lambda\mu\rangle\!\rangle=\sum_{\bar h\bar h'}U(k,\bar h^s,h,[10];\bar h,h^s)U(k',\bar h^s,h',[10];\bar h',h^s)$$
$$\times[P^{[10]\times[10]}(D^\dagger_{ij})\times|(\bar h\bar h'^*)\bar h^s\kappa\lambda\mu\rangle\!\rangle]^{h\times h'}_{h_1\times h'_1} \tag{5.8}$$

where $\bar h^s=[h^s_1-1,h^s_2]$ or $[h^s_1,h^s_2-1]$, the summations run over $\bar h=[\bar h_1\bar h_2]=[h_1-1,h_2]$, $[h_1,h_2-1]$, $\bar h'=[\bar h'_1\bar h'_2]=[h'_1-1,h'_2]$, $[h'_1,h'_2-1]$, and in the square bracket the U(2)× U(2) irreps $\bar h\times\bar h'$ and $[10]\times[10]$ are coupled to $h\times h'$. Equation (5.8) is the counterpart of the relation corresponding to the upper signs in equation (3.4). When we take the scalar product of both sides of equation (5.8) with an SU(2) coupled state and use the completeness of the latter set, we obtain the recursion relations sought for:

$$\left\langle\begin{matrix}h\\q\end{matrix},\begin{matrix}h'^*\\q'^*\end{matrix}\middle|\begin{matrix}\kappa\\\lambda\end{matrix};\ h^s\right\rangle$$
$$=\sum_{\bar h\bar h'}U(k,\bar h^s,h,[10];\bar h,h^s)U(k',\bar h^s,h',[10];\bar h',h^s)$$
$$\times\sum_{\bar q\bar q'}F(hq,h'q',\bar h\bar q,\bar h'\bar q',\lambda)\left\langle\begin{matrix}\bar h\\\bar q\end{matrix},\begin{matrix}\bar h'^*\\\bar q'^*\end{matrix}\middle|\begin{matrix}\kappa\\\lambda\end{matrix};\ \bar h^s\right\rangle \tag{5.9}$$

corresponding to $\bar h^s=[h^s_1-1,h^s_2]$ or $[h^s_1,h^s_2-1]$. Here $\bar q=[\bar q_1\bar q_2]$, $\bar q'=[\bar q'_1\bar q'_2]$ and

$$F(hq,h'q',\bar h\bar q,\bar h'\bar q',\lambda)$$
$$=\langle hq,h'^*q'^*,\lambda\|P^{[10]\times[10]}(D^\dagger_{ij})\|\bar h\bar q,\bar h'^*\bar q'^*,\lambda\rangle \tag{5.10}$$

is a reduced matrix element with respect to the U(2)×U(2) subgroup of U(2, 2).

The calculation of $F(hq,h'q',\bar h\bar q,\bar h'\bar q',\lambda)$, carried out in the appendix, is based upon the Wigner-Eckart theorem for both SU(2) and SU(3) (see e.g. Edmonds 1957, Hecht 1965). In the latter case, the only RWC needed are the fundamental ones, which are multiplicity free. Once normalised to unity, they are independent of the method used in their derivation (except for a possible overall phase factor). Since only the ratios of RWC corresponding to the same SU(3) irreps enter our formulae, the difference in normalisation and overall phase, if any, is irrelevant. We have therefore taken their explicit values from the work of Biedenharn and Louck (1968).

The result for $F(hq,h'q',\bar h\bar q,\bar h'\bar q',\lambda)$ can be written as

$$F(hq,h'q',\bar h\bar q,\bar h'\bar q',\lambda)=G(q,q',\bar q,\bar q',\lambda)H(hq,\bar h\bar q)H(h'q',\bar h'\bar q') \tag{5.11}$$

in terms of two functions $G(q,q',\bar q,\bar q',\lambda)$ and $H(hq,\bar h\bar q)$. In tables 1 and 2, the non-zero values of the latter are listed in terms of the increments

$$\Delta h_\alpha=h_\alpha-\bar h_\alpha\qquad \Delta h'_\alpha=h'_\alpha-\bar h'_\alpha\qquad \Delta q_\alpha=q_\alpha-\bar q_\alpha$$
$$\Delta q'_\alpha=q'_\alpha-\bar q'_\alpha\qquad \alpha=1,2. \tag{5.12}$$

Table 1. Non-zero values of the function $G(q, q', \bar{q}, \bar{q}', \lambda)$ in terms of the increments Δq_1, Δq_2, $\Delta q'_1$ and $\Delta q'_2$.

Δq_1	Δq_2	$\Delta q'_1$	$\Delta q'_2$	$G(q, q', \bar{q}, \bar{q}', \lambda)$
0	0	0	0	1
0	1	0	1	$-[(q'_1-q_2+\lambda_1+2)(q'_1-q_2+\lambda_2+1)]^{1/2}$
0	1	1	0	$-[(q_1-q'_1-\lambda_2+1)(q'_1-q_1+\lambda_1)]^{1/2}$
1	0	0	1	$-[(q'_1-q_1+\lambda_1+1)(q_1-q'_1-\lambda_2)]^{1/2}$
1	0	1	0	$[(q'_1-q_2+\lambda_1+1)(q'_1-q_2+\lambda_2)]^{1/2}$

Table 2. Non-zero values of the function $H(h, q, \bar{h}, \bar{q})$ in terms of the increments Δh_1, Δq_1, Δh_2 and Δq_2.

Δh_1	Δq_1	Δh_2	Δq_2	$H(h, q, \bar{h}, \bar{q})$
0	0	1	0	$[(q_1-h_2+1)(h_2-q_2)]^{1/2}$
1	0	0	0	$[(h_1-q_1)(h_1-q_2+1)]^{1/2}$
0	0	1	1	$[(q_1-h_2+1)(h_1-q_2+2)q_2]^{1/2}[(q_1-q_2+1)(q_1-q_2+2)]^{-1/2}$
1	0	0	1	$[(h_1-q_1)(h_2-q_2+1)q_2]^{1/2}[(q_1-q_2+1)(q_1-q_2+2)]^{-1/2}$
0	1	1	0	$-[(h_1-q_1+1)(h_2-q_2)(q_1+1)]^{1/2}[(q_1-q_2)(q_1-q_2+1)]^{-1/2}$
1	1	0	0	$[(q_1-h_2)(h_1-q_2+1)(q_1+1)]^{1/2}[(q_1-q_2)(q_1-q_2+1)]^{-1/2}$

It follows that the recursion relations (5.9) may contain up to twenty terms. In most cases of practical interest, however, only a few of them actually occur.

By iterating equation (5.9), any RWC

$$\left\langle \begin{matrix} h \\ q \end{matrix}, \begin{matrix} h'^* \\ q'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \lambda \end{matrix}; h^s \right\rangle$$

can be obtained from the stretched RWC

$$\left\langle \begin{matrix} k \\ \bar{q} \end{matrix}, \begin{matrix} k'^* \\ \bar{q}'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \lambda \end{matrix} \right\rangle$$

for which $\bar{h}^s = [\dot{0}]$ has been omitted since such RWC are multiplicity free.

Let us next consider the calculation of the D_{ij} matrix elements in the stretched state case. Our starting point is the relation

$$D_{ij}|(kk'^*)\kappa\lambda\mu\rangle = 0 \tag{5.13}$$

which is the counterpart of equation (3.9). By taking the scalar product of the Hermitian conjugate of equation (5.13) with an SU(2) coupled state, and by using the completeness of the latter set, we readily obtain

$$\sum_{qq'} F(kq, k'q', \bar{k}\bar{q}, \bar{k}'\bar{q}', \lambda) \left\langle \begin{matrix} k \\ q \end{matrix}, \begin{matrix} k'^* \\ q'^* \end{matrix} \middle| \begin{matrix} \kappa \\ q \end{matrix} \right\rangle = 0. \tag{5.14}$$

Regardless of whether $k_2 = 0$ or $k'_2 = 0$, equation (5.14) represents a set of two relations, each containing at most three terms. Whenever $k'_2 = 0$, for instance, for some fixed values of $\bar{q}$ and $\bar{q}'$, the two relations correspond to $\bar{k}\bar{k}' = [k_1, k_2-1][k'_1-1, 0]$ and $[k_1-1, k_2][k'_1-1, 0]$, respectively, while the summation runs over $qq' = [\bar{q}_1\bar{q}_2][\bar{q}'_1 0]$, $[\bar{q}_1+1, \bar{q}_2][\bar{q}'_1+1, 0]$, and $[\bar{q}_1, \bar{q}_2+1][\bar{q}'_1+1, 0]$.

From table 1, it follows that

$$G(q, q', \bar{q}, \bar{q}', \lambda) = G(q', q, \bar{q}', \bar{q}, \lambda^*). \tag{5.15}$$

Hence for fixed h^s, the coefficients in equations (5.9) and (5.14) remain invariant under the substitution $h \leftrightarrow h'$, $q \leftrightarrow q'$, $\kappa \to \kappa^*$, $\lambda \to \lambda^*$. The RWC therefore satisfy the symmetry relation

$$\left\langle \begin{matrix} h \\ q \end{matrix}, \begin{matrix} h'^* \\ q'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \lambda \end{matrix}; \; h^s \right\rangle = \left\langle \begin{matrix} h' \\ q' \end{matrix}, \begin{matrix} h^* \\ q^* \end{matrix} \middle| \begin{matrix} \kappa^* \\ \lambda^* \end{matrix}; \; h^s \right\rangle \tag{5.16}$$

which can be used to express the RWC with $k_2 = 0$ in terms of those with $k_2' = 0$. Consequently, we shall henceforth restrict ourselves to the latter.

The iteration of equation (5.14) is straightforward and leads to an expression of

$$\left\langle \begin{matrix} h \\ q \end{matrix}, \begin{matrix} k'^* \\ q'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \lambda \end{matrix} \right\rangle$$

in terms of

$$\left\langle \begin{matrix} k \\ \lambda \end{matrix}, \begin{matrix} k'^* \\ [\dot{0}] \end{matrix} \middle| \begin{matrix} \kappa \\ \lambda \end{matrix} \right\rangle.$$

By using the standard recursion relations (5.1), the latter can be obtained from the RWC

$$\left\langle \begin{matrix} k \\ k \end{matrix}, \begin{matrix} k'^* \\ [\dot{0}] \end{matrix} \middle| \begin{matrix} \kappa \\ k \end{matrix} \right\rangle = 1.$$

The complete result for the stretched RWC with $k_2' = \dot{0}$ is

$$\begin{aligned} \left\langle \begin{matrix} k \\ q \end{matrix}, \begin{matrix} k'^* \\ q'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \lambda \end{matrix} \right\rangle &= (-1)^{q_1-\lambda_1}[(k_1+1)!\,k_2!\,k_1'!\,(k_1-\lambda_1)!\,(k_1-\lambda_2+1)! \\ &\quad \times (k_2-\lambda_2)!\,(\lambda_1+k_1'+1)!\,(\lambda_2+k_1')!\,(q_1-k_2)!\,(\lambda_1-q_2)!\,(q_1-q_2+1)!]^{1/2} \\ &\quad \times [(k_1+k_1'+1)!\,(k_2+k_1')!\,(\lambda_1-k_2)!\,(k_1-q_1)!\,(k_1-q_2+1)!\,(k_2-q_2)! \\ &\quad \times (k_1'-q_1')!\,(q_1-\lambda_1)!\,(q_1-\lambda_2+1)!\,(q_2-\lambda_2)!\,(q_1+1)!\,q_2!]^{-1/2}. \end{aligned} \tag{5.17}$$

Equation (5.17) provides us with the starting coefficients for the iteration of equation (5.9). Except in low multiplicity cases, the latter is difficult, if not impossible, by an algebraic procedure. However, equation (5.9) is well suited to numerical computation. In this respect, it may be convenient to combine both recursion relations (5.1) and (5.9). Whenever λ is maximal, i.e. $\lambda = k$, equation (5.9) generally contains only a few terms and is therefore easily solved. Equation (5.1) can then be used to obtain the RWC corresponding to lower λ.

Before presenting a detailed example of the resolution of equation (5.9), in the next section we shall review various procedures for orthonormalising the SU(3) RWC.

6. Orthonormal SU(3) reduced Wigner coefficients

The SU(3) coupled states (4.9) are not normalised to unity, nor are they orthogonal with respect to h^s. Hence the same is true for the corresponding RWC. Orthonormal SU(3) coupled states and RWC can be obtained in various ways.

Following the Bargmann–Moshinsky procedure (1961) for defining orthonormal basis states for the U(3) irreps in the chain U(3) ⊃ SO(3), we can search for a Hermitian polynomial function X in the U(2, 2) generators which commutes with the U(2) × U(2) generators. Provided that X is independent of the Casimir operators of U(2, 2) and U(2) × U(2), and that its eigenvalues are distinct, its eigenvectors may serve as orthonormal basis states, characterised by the corresponding eigenvalues. A possible candidate for X is the operator

$$X=\sum_{i,j,k=1}^{2}[D^{\dagger}_{i,j+2}E_{ki}D_{k,j+2}+D^{\dagger}_{i,j+2}E_{k+2,j+2}D_{i,k+2}]. \tag{6.1}$$

In such a procedure, however, the interesting group theoretical meaning of the additional label h^s is completely lost.

Other methods make use of the overlap matrix of the non-orthonormal basis

$$t_{h^{s'},h^s}(h,h'^*,\kappa)=\langle\langle(hh'^*)h^{s'}\kappa\lambda\mu\,|\,(hh'^*)h^s\kappa\lambda\mu\rangle\rangle. \tag{6.2}$$

Such a matrix is obviously independent of the row $\lambda\mu$ of the U(3) irrep κ. It can be calculated either directly from the RWC determined in the previous section as follows:

$$t_{h^{s'},h^s}(h,h'^*,\kappa)=\sum_{qq'}\left\langle\begin{matrix}h\\ q\end{matrix},\begin{matrix}h'^*\\ q'^*\end{matrix}\,\middle|\,\begin{matrix}\kappa\\ \lambda\end{matrix};\ h^{s'}\right\rangle\left\langle\begin{matrix}h\\ q\end{matrix},\begin{matrix}h'^*\\ q'^*\end{matrix}\,\middle|\,\begin{matrix}\kappa\\ \lambda\end{matrix};\ h^{s}\right\rangle \tag{6.3}$$

or from the recursion relation

$$\begin{aligned}t_{h^{s'},h^s}(h,h'^*,\kappa)=&(h^s_1+h^s_2)^{-1}\sum_{\bar h^s\bar h^{s'}\bar h\bar h'}[\lambda(h^{s'},h,h')-\lambda(\bar h^{s'},\bar h,\bar h')]\\&\times\gamma(h^s,h^{s'},\bar h^s,\bar h^{s'})U(k,\bar h^{s'},h,[10];\bar h,h^{s'})U(k',\bar h^{s'},h',[10];\bar h',h^{s'})\\&\times U(k,\bar h^s,h,[10];\bar h,h^s)U(k',\bar h^s,h',[10];\bar h',h^s),\\&\times t_{\bar h^{s'},\bar h^s}(\bar h,\bar h'^*,\kappa)\end{aligned} \tag{6.4}$$

which was derived from a coherent state realisation of U(2, 2) (Quesne 1987b)†. In equation (6.4), the summations run over $\bar h^s=[h^s_1,h^s_2-1],[h^s_1-1,h^s_2]$, $\bar h^{s'}=[h^{s'}_1,h^{s'}_2-1]$, $[h^{s'}_1-1,h^{s'}_2]$, $\bar h=[h_1,h_2-1]$, $[h_1-1,h_2]$, $\bar h'=[h'_1,h'_2-1]$, $[h'_1-1,h'_2]$, and the coefficients λ and γ are, respectively, defined by

$$\begin{aligned}\lambda(h^s,h,h')=&\tfrac12[h_1(h_1+4)+h_2(h_2+2)+h'_1(h'_1+4)+h'_2(h'_2+2)\\&-h^s_1(h^s_1+3)-h^s_2(h^s_2+1)+9]\end{aligned} \tag{6.5}$$

and

$$\gamma(h^s,h^{s'},\bar h^s,\bar h^{s'})=(B_{\bar h^s}B_{\bar h^{s'}}/B_{h^s}B_{h^{s'}})^2 \tag{6.6}$$

where

$$(B_{h^s})^2=(h^s_1-h^s_2+1)[(h^s_1+1)!\,h^s_2!]^{-1}. \tag{6.7}$$

Since equation (6.4) is easily solved either algebraically or numerically, by comparison with equation (6.3) it provides us with a useful check of the values obtained for the RWC.

† The normalisation used by Quesne (1987b) differs from the current one, and so the former overlap matrix has to be multiplied by $(B_{h^s}B_{h^{s'}})^{-1}$ to obtain the overlap matrix (6.2).

Orthonormal SU(3) coupled states (distinguished from the non-orthonormal ones through the use of a curly bracket instead of an angular one) may be defined by (Deenen and Quesne 1984, Le Blanc and Rowe 1985, Quesne 1987b)

$$|(hh'^*)h^s\kappa\lambda\mu\} = \sum_{h^{s'}} |(hh'^*)h^{s'}\kappa\lambda\mu\rangle\!\rangle t^{-1/2}_{h^{s'},h^s}(h, h'^*, \kappa) \tag{6.8}$$

where $t^{-1/2}$ is the inverse square root of the positive definite Hermitian matrix t.

Alternatively, they may be obtained by a Gram-Schmidt orthonormalisation procedure similar to that applied by Vergados (1968), as well as by Draayer and Akiyama (1973). For such a purpose, let us enumerate the labels $h^{s(\varphi)}$, $\varphi = 1, 2, \ldots, m_{hh'^*\kappa}$, in the order of increasing values of h_1^s, i.e. $h_1^{s(\varphi)} < h_1^{s(\varphi+1)}$. Then the orthonormal SU(3) coupled states are defined by

$$|(hh'^*)\varphi\kappa\lambda\mu\} = \sum_{\varphi'=1}^{\varphi} |(hh'^*)\varphi'\kappa\lambda\mu\rangle\!\rangle C_{\varphi'\varphi} \tag{6.9}$$

where for simplicity we have substituted φ for $h^{s(\varphi)}$, and the coefficients $C_{\varphi'\varphi}$ are determined by the requirement

$$\{(hh'^*)\varphi'\kappa\lambda\mu|(hh'^*)\varphi\kappa\lambda\mu\} = \delta_{\varphi'\varphi}. \tag{6.10}$$

Since in most cases the overlap matrix (6.2) is close to a diagonal one, both prescriptions (6.8) and (6.9) have the advantage of approximately preserving the characterisation of states by h^s. The second one leading to simpler algebraic results in low-multiplicity cases will be used in the present paper. Hence we define the orthonormal SU(3) RWC by the following relation:

$$\left\langle \begin{matrix} h \\ q \end{matrix}, \begin{matrix} h'^* \\ q'^* \end{matrix} \middle| \begin{matrix} \kappa \\ \lambda \end{matrix}; \varphi \right\rangle = \langle hq, h'^*q'^*, \lambda\mu|(hh'^*)\varphi\kappa\lambda\mu\}. \tag{6.11}$$

It remains to fix the overall phase of the RWC. That of the non-orthonormal ones (4.16) is completely determined by the choice of signs in equations (4.6) and (4.12). The simplest and most natural way for fixing the phase of the orthonormal SU(3) coupled states, and consequently of the orthonormal SU(3) RWC, consists in defining $C_{\varphi\varphi}$ as a real and positive constant. We shall therefore adopt such a convention. This is different from the ordinary procedure, where a particular RWC is assigned to be positive for each mode of coupling, i.e. each φ. In the present case, the latter convention could of course be introduced *a posteriori*.

By working out a detailed example, in the next section we shall show that in low-multiplicity cases the resolution of the complementary recursion relations (5.9) combined with the prescription (6.9) leads to useful algebraic formulae for the RWC.

7. A detailed example

Let us calculate the RWC appearing in the matrix elements of U(3) adjoint tensor operators, i.e. those transforming under the U(3) irrep [1, 0, −1]. It is well known that such tensor operators are the first to exhibit a multiplicity problem, the irrep $[h_1h_20]$ being contained twice in $[h_1h_20]\times[1, 0, -1]$ whenever $h_1 > h_2 > 0$. Baird and Biedenharn (1964, 1965) solved this problem by defining two independent unit adjoint operators, respectively characterised by $\binom{0}{0\;\;0}$ or $\rho = 2$, and $\binom{0}{1\;\;-1}$ or $\rho = 1$. The first is proportional to the SU(3) generators, while the second is orthonormal to the latter. We shall therefore compare their RWC with the present ones.

Since, in SU(3), the irreps $[1, 0, -1]$ and $[0, -1, -2]$ are equivalent, we have to consider the product of U(3) irreps $[h_1 h_2 0] \times [0, -1, -2]$, containing $[h_1 - 1, h_2 - 1, -1]$ with multiplicity two if $h_1 > h_2 > 0$, and multiplicity one if either $h_1 = h_2 > 0$ or $h_1 > h_2 = 0$. Hence, in the present case, $h = [h_1 h_2]$, $h' = [2, 1]$, $k = [h_1 - 1, h_2 - 1]$ or $[h_1 - 1, 0]$, and $k' = [1, 0]$ or $[1, 1]$, according as $h_2 > 0$ or $h_2 = 0$. From King's branching rule, we find that $h^s = [1, 1]$ or $[2, 0]$ if $h_1 > h_2 > 0$, $h^s = [1, 1]$ if $h_1 = h_2 > 0$, and $h^s = [1, 0]$ if $h_1 > h_2 = 0$. As explained at the end of § 5, we shall restrict ourselves to the case of maximal λ, i.e. $\lambda = [h_1 - 1, h_2 - 1]$, where the complementary recursion relations (5.9) assume a very simple form.

In the multiplicity-two case, there are four SU(2) coupled states, corresponding to $qq'^* = [h_1 h_2][0, -2]$, $[h_1 h_2][-1, -1]$, $[h_1 h_2 - 1][0, -1]$ and $[h_1 - 1, h_2][0, -1]$, respectively, and hence eight RWC need to be determined. For each of them, equation (5.9) reduces to a single relation, associated with $\bar{h}^s = [1, 0]$ and containing only four terms (instead of the twenty it may comprise in general). As a result, the eight sought-for RWC are linear combinations of six RWC

$$\left\langle \begin{matrix} \bar{h} \\ \bar{q} \end{matrix}, \begin{matrix} \bar{h}'^* \\ \bar{q}'^* \end{matrix} \middle| \begin{matrix} \kappa \\ k \end{matrix}; \ [1, 0] \right\rangle$$

with

$$\bar{h}\bar{q}\bar{h}'^*\bar{q}'^* = [h_1, h_2 - 1, 0][h_1, h_2 - 1][0, 0, -2][0, -1], [h_1, h_2 - 1, 0]$$
$$[h_1 - 1, h_2 - 1][0, 0, -2][00], [h_1, h_2 - 1, 0][h_1, h_2 - 1]$$
$$\times [0, -1, -1][0, -1], [h_1 - 1, h_2, 0]$$
$$[h_1 - 1, h_2][0, 0, -2][0, -1], [h_1 - 1, h_2, 0][h_1 - 1, h_2 - 1][0, 0, -2][00]$$

and $[h_1 - 1, h_2, 0][h_1 - 1, h_2][0, 0, -1][0, -1]$, respectively. By now applying equation (5.9) to the latter, we can express them in terms of the single stretched RWC

$$\left\langle \begin{matrix} k \\ k \end{matrix}, \begin{matrix} k'^* \\ [\dot{0}] \end{matrix} \middle| \begin{matrix} \kappa \\ k \end{matrix} \right\rangle$$

equal to one. By combining both results, we obtain the analytical formulae given in table 3. For completeness, we also give there the RWC for the multiplicity-one cases. Note that the results for $h_1 = h_2 > 0$ can be obtained as a special case of those for $h_1 > h_2 > 0$.

From equation (6.3) and table 3, the overlap matrix $t(h, h'^*, \kappa)$ of the non-orthonormal basis becomes

$$t(h, h'^*, \kappa) = \begin{cases} \begin{pmatrix} 2h_1h_2 + 3h_1 + 5h_2 + 6 & -[3(h_1 - h_2)(h_1 - h_2 + 2)]^{1/2} \\ -[3(h_1 - h_2)(h_1 - h_2 + 2)]^{1/2} & 2h_1h_2 + h_1 + 3h_2 \end{pmatrix} & \text{if } h_1 > h_2 > 0 \\ (2(h_1 + 1)(h_1 + 3)) & \text{if } h_1 = h_2 > 0 \\ (h_1 + 3) & \text{if } h_1 > h_2 = 0. \end{cases} \tag{7.1}$$

As it should, it coincides with the solution of the recursion relation (6.4) obtained in a previous work (Quesne 1987b).

The orthonormal SU(3) RWC obtained from equations (6.9) and (6.11) are listed in table 4. For comparison, those of Baird and Biedenharn (1965) are given in table 5 in the multiplicity-two case (see also Louck and Biedenharn 1970); in the multiplicity-one cases, their RWC of course coincide (except for the phase) with the RWC of table 4.

Table 3. The non-orthonormal SU(3) RWC

$$\left\langle \begin{matrix}[h_1h_20] \\ q\end{matrix}, \begin{matrix}[0,-1,-2] \\ -q'^*\end{matrix} \middle| \begin{matrix}[h_1-1,h_2-1,-1] \\ [h_1-1,h_2-1]\end{matrix}; h' \right\rangle .$$

(a) $h_1>h_2>0$			
		h'	
q	q'^*	[11]	[20]
$[h_1h_2]$	$[0,-2]$	0	$[2(h_1+1)h_2]^{1/2}$
$[h_1h_2]$	$[-1,-1]$	$-[2(h_1+1)h_2]^{1/2}$	0
$[h_1,h_2-1]$	$[0,-1]$	$[3(h_1+1)(h_1-h_2+2)/(h_1-h_2+1)]^{1/2}$	$-[(h_1+1)(h_1-h_2)/(h_1-h_2+1)]^{1/2}$
$[h_1-1,h_2]$	$[0,-1]$	$[3h_2(h_1-h_2)/(h_1-h_2+1)]^{1/2}$	$[h_2(h_1-h_2+2)/(h_1-h_2+1)]^{1/2}$
(b) $h_1=h_2>0$			
		[11]	
$[h_1h_1]$	$[-1,-1]$	$-[2h_1(h_1+1)]^{1/2}$	
$[h_1,h_1-1]$	$[0,-1]$	$[6(h_1+1)]^{1/2}$	
(c) $h_1>h_2=0$			
		[10]	
$[h_10]$	$[0,-2]$	$\frac{1}{2}[3(h_1+2)]^{1/2}$	
$[h_10]$	$[-1,-1]$	$-\frac{1}{2}(h_1)^{1/2}$	
$[h_1-1,0]$	$[0,-1]$	$(\frac{3}{2})^{1/2}$	

8. Conclusion

In the present paper, we have shown that the solution to the SU(n) external state labelling problem proposed in I leads to two sets of recursion relations for the SU(n) RWC: the standard and the complementary ones. As a matter of fact, the demonstrations have been restricted to SU(2) and SU(3), but it is obvious that, at least in principle, they could be extended to higher SU(n) groups. For SU(2), the complementary recursion relations coincide with some relations previously found by Bargmann (1962), whereas, to the author's knowledge, for SU(3)—and in general SU(n)—they have not been encountered before.

Since both types of recursion relations result from the matrix elements of some group generators—either U(n) itself, or its U($n-1$, $n-1$) complementary group—from a group theoretical viewpoint they stand on an equal footing. This contrasts with other solutions to the SU(n) external state labelling problem, also based upon recursion relations different from the standard ones, such as the solutions proposed by Moshinsky (1963) and Baird and Biedenharn (1964). In the SU(3) case, the former employs an indirect method, wherein the RWC are calculated from some auxiliary coefficients, themselves obtained by solving some recursion relations (Brody *et al* 1965). For the latter, Draayer and Akiyama (1973) devised a recursive computational procedure,

Table 4. The orthonormal SU(3) RWC

$$\left\langle \begin{matrix}[h_1h_20] \\ q\end{matrix}, \begin{matrix}[0,-1,-2] \\ q'^*\end{matrix} \middle| \begin{matrix}[h_1-1,h_2-1,-1] \\ [h_1-1,h_2-1]\end{matrix}; \varphi \right\rangle$$

in terms of $A=[(h_1-h_2+1)(2h_1h_2+3h_1+5h_2+6)]^{-1/2}$ and $B=[2(h_1+3)(h_2+2)\times(h_1-h_2+1)(2h_1h_2+3h_1+5h_2+6)]^{-1/2}$.

(*a*) $h_1>h_2>0$

		φ	
q	q'^*	1	2
$[h_1h_2]$	$[0,-2]$	0	$(2h_1h_2+3h_1+5h_2+6)(h_1-h_1+1)^{1/2}B$
$[h_1h_2]$	$[-1,-1]$	$-[2(h_1+1)h_2(h_1-h_2+1)]^{1/2}A$	$-[3(h_1-h_2)(h_1-h_2+1)(h_1-h_2+2)]^{1/2}B$
$[h_1,h_2-1]$	$[0,-1]$	$[3(h_1+1)(h_1-h_2+2)]^{1/2}A$	$-(h_1+4)[2h_2(h_1-h_2)]^{1/2}B$
$[h_1-1,h_2]$	$[0,-1]$	$[3h_2(h_1-h_2)]^{1/2}A$	$(h_2+3)[2(h_1+1)(h_1-h_2+2)]^{1/2}B$

(*b*) $h_1=h_2>0$

		1
$[h_1h_1]$	$[-1,-1]$	$-[h_1/(h_1+3)]^{1/2}$
$[h_1,h_1-1]$	$[0,-1]$	$[3/(h_1+3)]^{1/2}$

(*c*) $h_1>h_2=0$

		1
$[h_10]$	$[0,-2]$	$\frac{1}{2}[3(h_1+2)/(h_1+3)]^{1/2}$
$[h_10]$	$[-1,-1]$	$-\frac{1}{2}[h_1/(h_1+3)]^{1/2}$
$[h_1-1,0]$	$[0,-1]$	$3^{1/2}[2(h_1+3)]^{-1/2}$

Table 5. Baird and Biedenharn orthonormal SU(3) RWC

$$\left\langle \begin{matrix}[h_1h_20] \\ q\end{matrix}, \begin{matrix}[0,-1,-2] \\ q'^*\end{matrix} \middle| \begin{matrix}[h_1-1,h_2-1,-1] \\ [h_1-1,h_2-1]\end{matrix}; \rho \right\rangle$$

for $h_1>h_2>0$ in terms of $A=\frac{1}{2}(h_1-h_2+1)(h_1^2-h_1h_2+h_2^2+3h_1)^{-1/2}$ and $B=\frac{1}{2}(h_1+3)\times(h_2+2)(h_1-h_2+1)(h_1^2-h_1h_2+h_2^2+3h_1)^{-1/2}$.

		ρ	
q	q'^*	2	1
$[h_1h_2]$	$[0,-2]$	$-[3(h_1-h_2)(h_1-h_2+1)\times(h_1-h_2+2)]^{1/2}A$	$(h_1+h_2+6)[(h_1+1)h_2(h_1-h_2+1)]^{1/2}B$
$[h_1h_2]$	$[-1,-1]$	$(h_1+h_2)(h_1-h_2+1)^{1/2}A$	$[3(h_1+1)h_2(h_1-h_2)(h_1-h_2+1)\times(h_1-h_2+2)]^{1/2}B$
$[h_1,h_2-1]$	$[0,-1]$	$-[6h_2(h_1-h_2+2)]^{1/2}A$	$-(2h_1-h_2+6)[2(h_1+1)(h_1-h_2)]^{1/2}B$
$[h_1-1,h_2]$	$[0,-1]$	$-[6(h_1+1)(h_1-h_2)]^{1/2}A$	$-(h_1-2h_2-3)[2h_2(h_1-h_2+2)]^{1/2}B$

where the RWC are determined from parent ones by combining the characteristic null space properties of the SU(3) RWC (see e.g. Biedenharn *et al* 1985) with the knowledge of the SU(3) generator matrix elements.

The greater simplicity and elegance of the present approach have been obtained at the cost of the SU(3) coupled state and RWC non-orthonormality. This however cannot be regarded as a drawback of the theory since in other works, either orthonormalisation (Draayer and Akiyama 1973) or diagonalisation (Brody *et al* 1965) procedures are also required.

As has been shown for SU(3) in a detailed example, besides its interesting group theoretical properties, the present solution to the SU(n) external state labelling problem is also quite practical for numerical purposes. A simple algorithm, combining both sets of recursion relations, has been outlined. Its basic ingredients are the stretched RWC. Hence it seems ideally suited to deal with the cases where some of the irreps labels are very large. This is a situation where, even in the SU(2) case, some computational problems may arise. In the present construction, by acting with the complementary recursion relations, the large labels may be decreased until one arrives at a stretched coupling. The validity of this observation remains of course to be checked in practical cases. For such a purpose, one would have to develop a computer code based on the algorithm proposed in this paper. Such a code would then provide an alternative to the presently available one (Akiyama and Draayer 1973).

From a more fundamental viewpoint, the present work raises two interesting questions we hope to answer in forthcoming publications. The first one is whether there exists any relation between the Baird and Biedenharn solution (1964) and the present one. Although tables 4 and 5 show that they lead to different sets of RWC, some recent results on the SU(3) tensor operator structure (Biedenharn and Flath 1984, Bracken and MacGibbon 1984, Deenen and Quesne 1986) hint at possible connections between both solutions. The second question is whether the generating function techniques, devised by Schwinger (1965) and Bargmann (1962) to derive the SU(2) WC properties, and later on extended to SU(3) by Resnikoff (1967) and Hage Hassan (1983), can be exploited to deal with the SU(3) RWC as defined in the present paper.

Appendix. Calculation of $F(hq, h'q', \bar{h}\bar{q}, \bar{h}'\bar{q}', \lambda)$

The purpose of this appendix is to find the explicit form of $F(hq, h'q', \bar{h}\bar{q}, \bar{h}'\bar{q}', \lambda)$, defined in equation (5.10).

By applying the Wigner–Eckart theorem with respect to the U(2) × U(2) subgroup of U(2, 2), F can be written as

$$\begin{aligned} F(hq, h'q', \bar{h}\bar{q}, \bar{h}'\bar{q}', \lambda) &= \langle hq, h'q', \lambda | P^{[10]\times[10]}_{m\times m'}(D^{\dagger}_{ij}) | \bar{h}\bar{q}, \bar{h}'\bar{q}', \lambda\rangle \\ &\quad \times [\langle \tfrac{1}{2}(\bar{h}_1-\bar{h}_2)\, \tfrac{1}{2}(\bar{h}_1-\bar{h}_2), \tfrac{1}{2}\, m - \tfrac{1}{2} | \tfrac{1}{2}(h_1-h_2)\, \tfrac{1}{2}(h_1-h_2)\rangle \\ &\quad \times \langle \tfrac{1}{2}(\bar{h}'_1-\bar{h}'_2)\, \tfrac{1}{2}(\bar{h}'_1-\bar{h}'_2), \tfrac{1}{2}\, m' - \tfrac{1}{2} | \tfrac{1}{2}(h'_1-h'_2)\, \tfrac{1}{2}(h'_1-h'_2)\rangle]^{-1} \end{aligned} \tag{A1}$$

in terms of two SU(2) WC and of the matrix element of $P^{[10]\times[10]}_{m\times m'}(D^{\dagger}_{ij})$, where $m = h_1 - \bar{h}_1$ and $m' = h'_1 - \bar{h}'_1$, between two SU(2) coupled states of highest weight with respect to U(2) × U(2). The problem therefore amounts to determining the latter.

From equations (4.4) and (4.7), it follows that

$$P^{[10]\times[10]}_{m\times m'}(D^{\dagger}_{ij}) = [T^{[100]}_{[00]}(m) \times U^{[00-1]}_{[00]}(m')]^{[00]}_0 - \sqrt{2}[T^{[100]}_{[10]}(m) \times U^{[00-1]}_{[0-1]}(m')]^{[00]}_0 \tag{A2}$$

where

$$T^{[100]}_{[10]1}(m) = \eta_{2-m,1} \qquad T^{[100]}_{[10]0}(m) = \eta_{2-m,2}$$
$$T^{[100]}_{[00]0}(m) = \eta_{2-m,3} \qquad m = 1, 0 \tag{A3}$$

and

$$U^{[00-1]}_{[00]0}(m') = \eta_{4-m',3} \qquad U^{[00-1]}_{[0-1]0}(m') = -\eta_{4-m',2}$$
$$U^{[00-1]}_{[0-1]-1}(m') = \eta_{4-m',1} \qquad m' = 1, 0 \tag{A4}$$

are the components of irreducible tensors with respect to the first or the second U(3) group in the product U(3)×U(3), respectively. In equation (A2), the square bracket denotes a coupling of U(2) irreps.

By combining equation (A2) with standard U(2) recoupling techniques, we obtain the following result

$$\begin{aligned}\langle hq, h'q', \lambda | P^{[10]\times[10]}_{m\times m'}(D^{\dagger}_{ij}) | \bar{h}\bar{q}, \bar{h}'^{*}\bar{q}'^{*}, \lambda\rangle \\ &= \delta_{\bar{q}q}\delta_{\bar{q}'q'}\langle hq \| T^{[100]}_{[00]}(m) \| \bar{h}q\rangle\langle h'^{*}q'^{*} \| U^{[00-1]}_{[00]}(m') \| \bar{h}'^{*}q'^{*}\rangle \\ &\quad + [(q'_1 - q'_2 + 1)/(\bar{q}'_1 - \bar{q}'_2 + 1)]^{1/2} U(\lambda^{*}, \bar{q}, q', [10]; \bar{q}', q) \\ &\quad \times \langle hq \| T^{[100]}_{[10]}(m) \| \bar{h}\bar{q}\rangle\langle h'^{*}q'^{*} \| U^{[00-1]}_{[0-1]}(m') \| \bar{h}'^{*}\bar{q}'^{*}\rangle\end{aligned} \tag{A5}$$

in terms of a U(2) recoupling coefficient and of reduced matrix elements with respect to the U(2) subgroup of U(3)⊂U(3)×U(3). The Wigner–Eckart theorem with respect to U(3) enables to write the latter as

$$\begin{aligned}\langle hq \| T^{[100]}_{\sigma}(m) \| \bar{h}\bar{q}\rangle &= [\langle \tfrac{1}{2}(\bar{h}_1 - \bar{h}_2)\ \tfrac{1}{2}(\bar{h}_1 - \bar{h}_2), \tfrac{1}{2}\ m - \tfrac{1}{2} | \tfrac{1}{2}(h_1 - h_2)\ \tfrac{1}{2}(h_1 - h_2)\rangle]^{-1} \\ &\quad \times \left[\left\langle \begin{matrix}\bar{h} \\ \bar{q}\end{matrix}, \begin{matrix}[100] \\ \sigma\end{matrix} \middle| \begin{matrix} h \\ q\end{matrix}\right\rangle \left(\left\langle \begin{matrix}\bar{h} \\ \bar{h}\end{matrix}, \begin{matrix}[100] \\ [10]\end{matrix} \middle| \begin{matrix} h \\ h\end{matrix}\right\rangle\right)^{-1}\right] \\ &\quad \times \langle h(\max) | T^{[100]}_{[10]m}(m) | \bar{h}(\max)\rangle\end{aligned} \tag{A6}$$

and

$$\begin{aligned}\langle h'^{*}q'^{*} \| U^{[00-1]}_{\sigma^{*}}(m') \| \bar{h}'^{*}\bar{q}'^{*}\rangle \\ &= [\langle \tfrac{1}{2}(\bar{h}'_1 - \bar{h}'_2)\ -\tfrac{1}{2}(\bar{h}'_1 - \bar{h}'_2), \tfrac{1}{2} - m' + \tfrac{1}{2} | \tfrac{1}{2}(h'_1 - h'_2)\ -\tfrac{1}{2}(h'_1 - h'_2)\rangle]^{-1} \\ &\quad \times \left[\left\langle \begin{matrix}\bar{h}'^{*} \\ \bar{q}'^{*}\end{matrix}, \begin{matrix}[00-1] \\ \sigma^{*}\end{matrix} \middle| \begin{matrix} h'^{*} \\ q'^{*}\end{matrix}\right\rangle \left(\left\langle \begin{matrix}\bar{h}'^{*} \\ \bar{h}'^{*}\end{matrix}, \begin{matrix}[00-1] \\ [0-1]\end{matrix} \middle| \begin{matrix} h'^{*} \\ h'^{*}\end{matrix}\right\rangle\right)^{-1}\right] \\ &\quad \times \langle h'^{*}(\min) | U^{[00-1]}_{[0-1]-m'}(m') | \bar{h}'^{*}(\min)\rangle\end{aligned} \tag{A7}$$

where σ is either [10] or [00].

By using equation (4.12), the matrix elements on the right-hand side of equations (A6) and (A7) can be easily calculated and are

$$\begin{aligned}\langle h(\max) | T^{[100]}_{[10]m}(m) | \bar{h}(\max)\rangle &= \langle h^{*}(\min) | U^{[00-1]}_{[0-1]-m}(m) | \bar{h}^{*}(\min)\rangle \\ &= \begin{cases} [(h_1+1)(h_1-h_2)/(h_1-h_2+1)]^{1/2} & \text{if } \bar{h} = [h_1 - 1, h_2, 0],\ m = 1 \\ [h_2(h_1-h_2+1)/(h_1-h_2+2)]^{1/2} & \text{if } \bar{h} = [h_1, h_2 - 1, 0],\ m = 0. \end{cases}\end{aligned} \tag{A8}$$

To obtain equation (5.11) and the results of tables 1 and 2, it only remains to introduce equations (A5)–(A8) into equation (A1) and to replace the SU(2) coupling and recoupling coefficients, as well as the SU(3) RWC, by their explicit values.

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